Chapter 18.3

Summary

- We have studied several "Fundamental Theorems"

each of type:

Integral of a derivative on an oriented domain	=	Integral over the <u>oriented boundary</u> of the domain

- FTC

$$\underbrace{\int_a^b f'(x)\,dx}_{\text{integral of derivative over } [a,b]} = \underbrace{f(b) - f(a)}_{\text{"integral" over the oriented boundary of } [a,b]}$$

- Fundamental theorem for line integrals

$$\underbrace{\int_C \nabla f \cdot dr}_{\text{Integral of derivative over a curve}} = \underbrace{f(Q) - f(P)}_{\text{"integral" over the boundary } \partial C = Q - P}$$

- **Green's thm**

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_C F\cdot dr$$

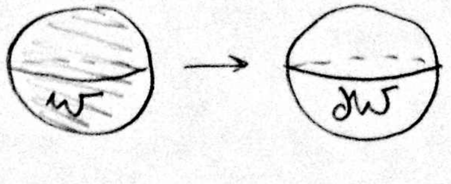
- **Stokes' thm**

$$\underbrace{\iint_S \operatorname{curl}(F)\cdot dS}_{\text{integral of derivative over a surface}} = \underbrace{\int_C F\cdot dr}_{\text{Integral over boundary curve}}$$

- **Divergence theorem**

$$\underbrace{\iiint_W \operatorname{div}(F)\, dV}_{\text{integral of derivative over 3D region}} = \underbrace{\iint_S F\cdot dS}_{\text{integral over boundary surface}}$$

$S \leftarrow$ closed, normal outward.



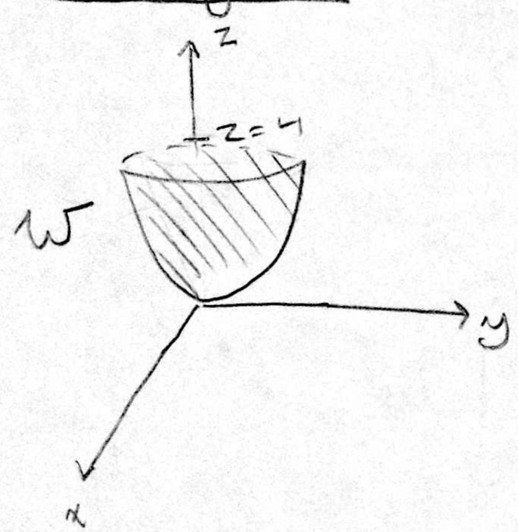
Chapter 18.3 Additional exercises

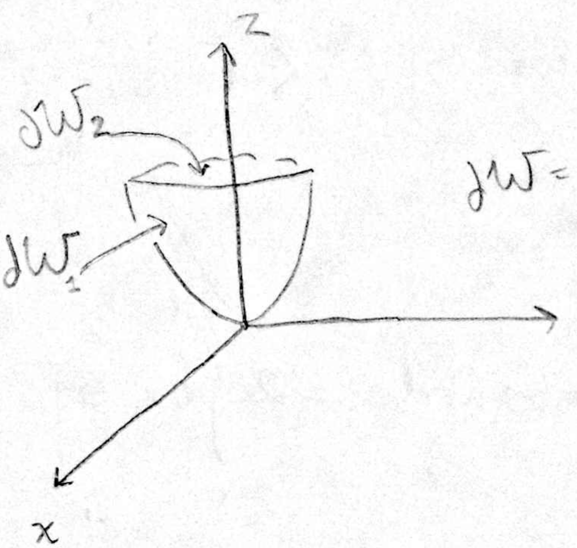
. Verify the Divergence Theorem for the vector field and region: $F(x,y,z) = \langle x,0,0\rangle$, $\underbrace{x^2+y^2 \le z \le 4}_{W}$.

we need to show:

$$\underbrace{\iint_{\partial W} F \cdot dS}_{(1)} = \underbrace{\iiint_{W} \operatorname{div}(F)\, dW}_{(2)}$$

Computing (1):





$\partial W = \partial W_1 \cup \partial W_2$

∂W_1 can be parametrized as follows:

$G(\theta,t) = (t\cos\theta,\ t\sin\theta,\ t^2) \quad 0 \le \theta \le 2\pi,\ 0 \le t \le 2$

$G_\theta(\theta,t) = (-t\sin\theta,\ t\cos\theta,\ 0)$

$G_t(\theta,t) = (\cos\theta,\ \sin\theta,\ 2t)$

$$N = G_\theta \times G_t = \begin{vmatrix} i & j & k \\ -t\sin\theta & t\cos\theta & 0 \\ \cos\theta & \sin\theta & 2t \end{vmatrix}$$

$$= \langle 2t^2\cos\theta,\ 2t^2\sin\theta,\ -t\rangle.$$

Checking if it is outward pointing:

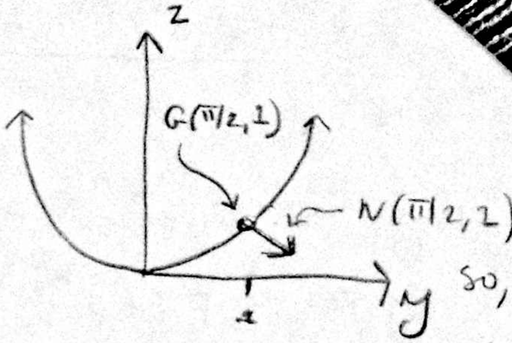
look at projection

$x=0$, i.e. $\theta = \pi/2$

Then, $G(\pi/2, t) = \langle 0, t, t^2 \rangle$

$N(\pi/2, t) = \langle 0, 2t^2, -t \rangle$

$N(\pi/2, 1) = \langle 0, 2, -1 \rangle$



So, points outward as needed.

$$\text{So, } \iint_{\partial W_1} F \cdot dS = \int_0^{2\pi}\int_0^2 F(G(\theta,t)) \cdot N(\theta,t)\, dt\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \langle t\cos\theta, 0, 0 \rangle \cdot \langle 2t^2\cos\theta, 2t^2\sin\theta, -t \rangle\, dt\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 2t^3\cos^2\theta\, dt\, d\theta = 2\int_0^{2\pi} \cos^2\theta \left[\frac{t^4}{4}\Big|_0^2 \right] d\theta$$

$$= 8\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{8}{2}\int_0^{2\pi} 1 + \cos 2\theta\, d\theta$$

$$= \boxed{8\pi}$$

↳

∂W_2 a disk: $\displaystyle\iint_{\partial W_2} F \cdot ds = \iint_{\partial W_2} \langle x,0,0\rangle \cdot \langle 0,0,1\rangle \, dS$

$= 0.$

So, $\displaystyle\iint_{\partial W} F \cdot dS = 8\pi.$

<u>Computing (2)</u>:

$\operatorname{div}(F) = 1$. The region W is cylindrical coordinates is:

$r^2 \le z \le 4$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

So, $\displaystyle\iiint_W 1 \, dV = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^{2} 4r - r^3 \, dr \, d\theta$

$\displaystyle= \int_0^{2\pi} 2r^2 - \frac{r^4}{4} \Big|_0^2 \, d\theta = \int_0^{2\pi} 4 \, d\theta = 2 \cdot 4\pi = \boxed{8\pi}$ ✓

2. Use the divergence theorem to evaluate the flux $\iint_S F\cdot dS$ of $F(x,y,z) = \langle x, y^2, z+y\rangle$, S is the boundary of the region contained in the cylinder $x^2+y^2=4$ between the planes $z=x$ and $z=8$.

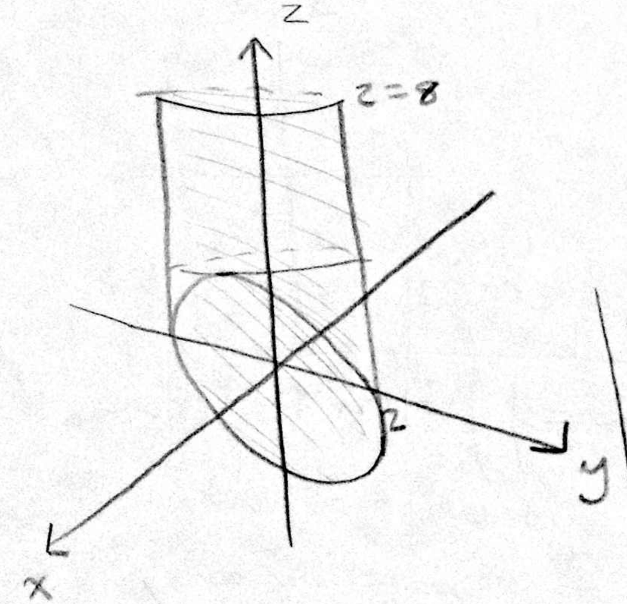
The divergence theorem tells us that:

$$\iint_S F\cdot dS = \iiint_W \operatorname{div}(F)\,dV = \iiint_W 1+2y+1 \; dV$$

$W \leftarrow$ region enclosed by S

$$= 2\iiint_W (1+y)\,dV.$$

Sketch region:



cylindrical coordinates

$x \le z \le 8 \iff r\cos\theta \le z \le 8$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

So, $2\iiint_W (1+y)\,dV =$

$$2\int_0^{2\pi}\int_0^{2}\int_{r\cos\theta}^{8} (1+r\sin\theta)\,r\,dz\,dr\,d\theta =$$

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$$2\int_0^{2\pi}\int_0^{2}\int_{r\cos\theta}^{8} r + r^2\sin\theta \, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^{2} rz + r^2 z\sin\theta \Big|_{r\cos\theta}^{8} \, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^{2} 8r + 8r^2\sin\theta - r^2\cos\theta - r^3\cos\theta\sin\theta \, dr\, d\theta$$

$$= 2\int_0^{2\pi} \frac{8r^2}{2} + \frac{8r^3}{3}\sin\theta - \frac{r^3}{3}\cos\theta - \frac{r^4}{4}\cos\theta\sin\theta \, dr\, d\theta \Big|_0^{2}$$

$$= 2\int_0^{2\pi} 16 + \frac{8\cdot 8}{3}\sin\theta - \frac{8}{3}\cos\theta - 4\cos\theta\sin\theta \, d\theta$$

$$= 64\pi + \frac{8\cdot 8\cdot 2}{3}\cancelto{0}{\int_0^{2\pi}\sin\theta\, d\theta} - \frac{2\cdot 8}{3}\cancelto{0}{\int_0^{2\pi}\cos\theta\, d\theta} - 8\cancelto{0}{\int_0^{2\pi}\cos\theta\sin\theta\, d\theta}$$

$$= \boxed{64\pi}$$